ON THREE-DIMENSIONAL NON-LINEAR EFFECTS IN THE STABILITY OF PARALLEL FLOWS

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Summary—The laminar motion to be examined is two-dimensional Poiseuille flow between parallel planes; this flow exhibits some of the important features of boundary-layer instability but has no property analogous to steady boundary-layer growth.

A two-dimensional Tollmien-Schlichting wave of given streamwise wave-length, α, is allowed to interact with a three-dimensional wave of the same α. The two-dimensional wave is of the form $A\,(t)\exp\,(i\alpha x)$, and the three-dimensional wave is represented by $B\,(t)\exp\,(i\alpha x)\cos\beta y$, where t is the time, x the streamwise distance and y the cross-stream distance. The non-linear interaction of the amplitudes A and B is described together with the various three-dimensional components of flow generated by this interaction.

1. INTRODUCTION

IN recent work on the stability of parallel flows, both the second-order three-dimensional aspects (Lin and Benney[1], Benney[2]) and the non-linear effects on amplitude (Stuart[3], Watson[4]) have received attention; it is the object of the present paper to present a unified theory possessing the essential features of this earlier work, together with some new aspects, by considering the non-linear interaction of two- and three-dimensional disturbances. In order to develop the theory as precisely as possible, attention will be focused mainly on the case of Poiseuille flow between parallel planes. This flow possesses the mathematical advantage that it is a purely parallel flow, which does not vary in the stream direction and has constant local Reynolds number; by comparison with experiment this is a disadvantage, since boundary-layer flows are not strictly parallel and one may expect the growth of the boundary layer to play an important role in the growth and development of disturbances. In the Blasius case of flow in a zero pressure gradient this effect is mainly one of increase of local Reynolds number, since the velocity profile remains of the same shape. By restricting attention to a purely parallel case, therefore, certain features are omitted but it is to be expected that many important features of the non-linear equations will be present in this simpler case. However, it is to be emphasized that this paper, as with previous work, is a perturbation theory and cannot be expected to have validity for amplitudes which are too large.

The present work is based on the full equations of motion (the Navier–Stokes equations) for Reynolds numbers such that the rate of amplification or decay of disturbances of linearized theory is sufficiently small. Disturbances to the basic laminar motion are considered to amplify in time; this follows linearized theory (e.g. Lin(5)) and the non-linear work of refs. 1–4, and this feature is retained for mathematical simplicity. However Watson(6) has shown how both linear and non-linear instability theories may be modified to treat the case of disturbances which grow in the streamwise direction instead of in time, provided that the rate of amplification or decay is sufficiently small. Basically his analysis is similar to that of refs. 3, 4, though it is algebraically more complicated.

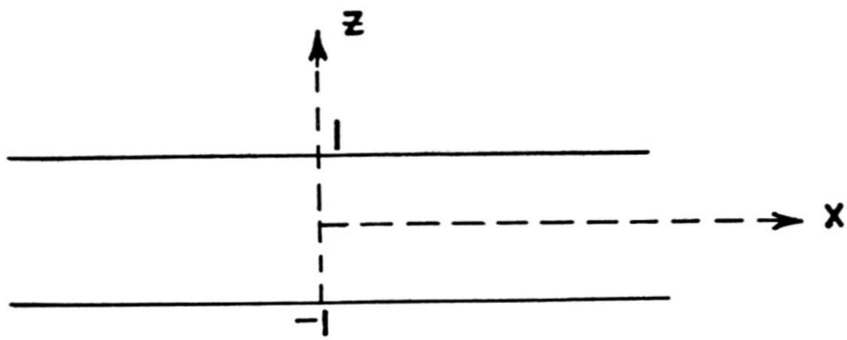

FIG. 1. Co-ordinate system, Y is spanwise.

It is perhaps permissible to remind the reader of a few of the essential features of the linearized theory, and of the non-linear developments which have so far been reported. If $\bar{u}_l \equiv 1-z^2$ denotes the basic Poiseuille flow, where z is the dimensionless distance between the planes at $z = \pm 1$ (Fig. 1), then a three-dimensional velocity fluctuation may be superimposed on the laminar flow and may be assumed to be proportional to $\exp [i(\alpha x+\beta y-\alpha ct)]$. (This is called a three-dimensional Tollmien–Schlichting wave.) In the above expression x is the streamwise co-ordinate, y the spanwise co-ordinate and t the time, while $2\pi/\alpha$ and $2\pi/\beta$ are streamwise and spanwise wave-lengths respectively. It is normally assumed that α and β are real, but that c is complex; the question of stability or instability is then decided by the imaginary part of c, namely c_i. If c_i is positive the Tollmien–Schlichting disturbance increases exponentially in time like $\exp(\alpha c_i t)$, and the flow is said to be unstable according to linearized instability theory; if c_i is necessarily negative the flow is stable, and if c_i

is zero the flow is neutrally stable. (The modification by Watson in ref. 6 considers α complex, β zero and the frequency, αc, real; stability or instability is then determined by the sign of α_i, the imaginary part of α). The sign of c_i naturally depends on the values of α, β and R, and this dependence is illustrated schematically in Fig. 2 for $\beta \equiv 0$. The neutral curve in the α, R plane separates regions of $c_i > 0$ and $c_i < 0$. The extension of this picture to the case of a three-dimensional neutral surface (α,β,R) may be made by means of the work of Squire[7] and the extension

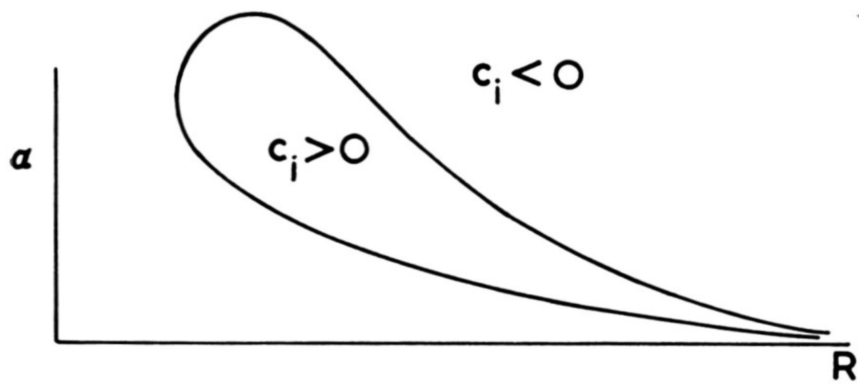

FIG. 2. Schematic neutral curve.

of Watson[8]. It suffices to say here that a three-dimensional disturbance ($\beta \neq 0$) at a given Reynolds number is equivalent to a two-dimensional disturbance ($\beta \equiv 0$) at a lower Reynolds number but with a higher value of α. For given values of α, R and c—the eigenvalues—it is possible to calculate the velocity distributions of the three components of a disturbance. Information of this kind gives the point of departure for a study of the non-linear aspects, some of which will now be described.

In the papers of Stuart[3] and Watson[4] the question discussed is that of the possible effects of the non-linear terms on the growth of the amplitude of a two-dimensional disturbance ($\beta \equiv 0$), and on its frequency and wave speed. It appears that an important effect is that, at least for sufficiently small values of c_i, the wave disturbance grows exponentially in time only for small amplitudes; at longer amplitudes it may equilibrate into an oscillation of a definite, finite amplitude which is independent of the initial amplitude. (This equilibrating-amplitude property is found experimentally in the laminar wake of a body, for example; the oscillatory wake arises from an instability, as shown by Tritton[9] for a certain Reynolds number range, but remains as a regular oscillation without producing turbulence). A second possibility is that a disturbance may grow at Reynolds numbers below the critical, provided its amplitude lies above a threshold value. (This would correspond to the result obtained earlier by Meksyn and Stuart[10].) These two possibilities are illustrated in Fig. 3, which shows (a) the case of a disturbance, the square of whose amplitude, $|A|^2$, tends to an equilibrium finite value as $t \to +\infty$ (this is sometimes described

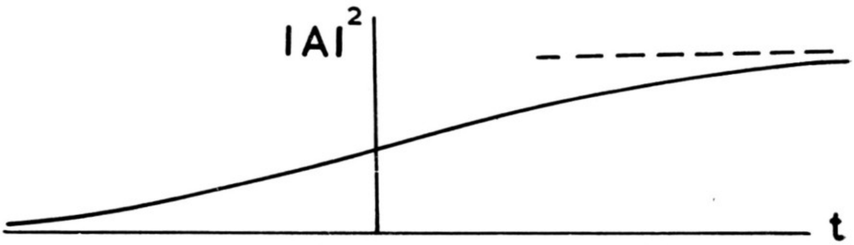

FIG. 3(a). Amplitude growth of two-dimensional oscillation in supercritical case.

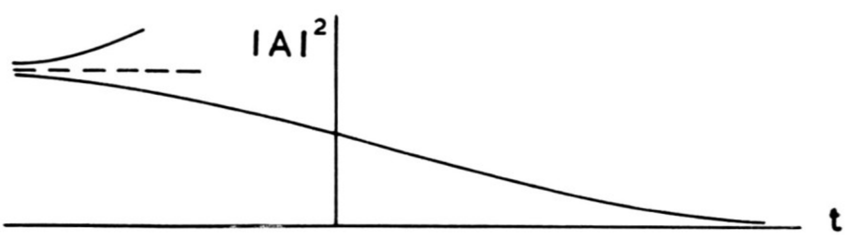

FIG. 3(b). Amplitude variation of two-dimensional oscillation in subcritical case.

as the supercritical case because the Reynolds number lies above the critical); and (b) a disturbance whose amplitude increases only if it lies above a threshold value (this is sometimes described as the subcritical case because the Reynolds number lies below the critical). Detailed calculations, which are not yet completed, are required to determine which possibility occurs in a given range of Reynolds number and wave number.

The non-linear features which control the amplification in the work described above are (i) the distortion of the mean motion by the Reynolds stress of the fundamental (Tolmien–Schlichting) wave, (ii) the generation of the first harmonic of the fundamental wave, $\exp[2i\alpha(x-c_r t)]$, where c_r is the real part of c, and (iii) the distortion of the fundamental wave. It can be shown that higher harmonics are of much smaller order of magnitude (even in the critical layer), and that the mean-motion distortion and the first harmonic component are of similar order of magnitude. The requirement that the motion be of this type, with the energy disturbance concentrated in the fundamental wave and one harmonic, appears to be that $c_i \ll (\alpha R)^{-1/3}$, where c_i is the imaginary part of c and $(\alpha R)^{-1/3}$ is the dimensionless thickness of the "critical" layer. In order to calculate the basic solution of the type described above, it is necessary to go at least to the third order in amplitude.

The work described by Lin and Benney[1, 2] in two very important papers is concerned with the problem of the interaction of a two-dimensional wave disturbance, $\exp(i\alpha(x-c_1t))$, with a three-dimensional wave disturbance, $\exp(i\alpha(x-c_2t)) \cos \beta y$, to the second order in amplitude. It will be noticed in these exponential forms that the complex numbers c_1 and c_2 are in general different; although they recognize this fact, Lin and Benney assume for mathematical simplicity that $c_1 = c_2$ at a given Reynolds number, because in Blasius boundary-layer flow the real parts of c_1 and c_2 (c_{1r} and c_{2r}, the wave speeds) appear to differ at most by about 15%. The difference in c_{1r} and c_{2r} means that one important component of the flow generated by the non-linear interaction of the two fundamentals may have a frequency of about $^1/_6$th or $^1/_7$th of the fundamental frequencies; we shall return to a discussion of this in Section 4.

Lin and Benney point out that the interaction of the two fundamentals produced second-order effect of several kinds, including the generation of harmonics of the fundamental modes, the modification of the original mean motion and (with the approximation $c_1 = c_2$) the generation of new harmonic components of flow which are non-periodic in time. It is with the calculation of the latter flows that Benney's paper is mainly concerned. There are two such flows; the first is proportional to $\cos 2\beta y$ (except for the spanwise velocity, which is proportional to $\sin 2\beta y$), is independent of x, and is exactly non-periodic in t even when $c_1 \neq c_2$. The second flow is non-periodic because of the approximation $c_1 = c_2$ and is proportional to $\cos \beta y$ (except for the spanwise velocity, which is proportional to $\sin \beta y$). The important feature of these harmonic flow components, as Benney's analysis shows, is that they lead to a spanwise transfer of energy, a feature which has been observed experimentally by Klebanoff and Tidstrom[11, 12]. Both the $\cos 2\beta y$ and $\cos \beta y$ flows possess streamwise vorticity, and Lin and Benney argue that the latter, which has the same spanwise wave-length as the three-dimensional fundamental, is likely to be the more important in experiment. The argument for this is that the wave is initially two-dimensional, so that the three-dimensional fundamental is smaller in amplitude than the two-dimensional one; consequently the $\cos \beta y$ term, which arises from a direct interaction of the two fundamentals, is larger than the $\cos 2\beta y$ term, which arises as a harmonic of the three-dimensional (and smaller) fundamental alone.

In Sections 2 and 3 of this paper an analysis will be described which incorporates both the non-linear effects on the amplitude[3,4], and the three-dimensional effects[1,2], in the solution of the Navier–Stokes equations, while a discussion of the nature of the solution appears in Section 4.

2. FOURIER ANALYSIS OF THE EQUATIONS OF MOTION

We consider the flow between two parallel planes, set at $z = \pm 1$, and further we let x denote the co-ordinate in the flow direction and y the transverse co-ordinate (see Fig. 1). Velocity components corresponding to x, y, z are denoted by u, v, w, while p denotes pressure and t the time. Quantities have been made dimensionless with a reference length h, the half-distance between the planes, a reference speed U_0, the maximum speed in the basic laminar flow, a reference time h/U_0, and a reference pressure ϱU_0^2, ϱ being the density. The Reynolds number is $R = U_0 h/\nu$.

The Navier-Stokes and continuity equations for incompressible flow may be written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{R}\nabla^2 u \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{R}\nabla^2 v \tag{2.2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w \tag{2.3}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{2.4}$$

The basic flow whose stability we wish to examine is given by

$$u = \bar{u} = 1 - z^2, \quad v = w = 0, \quad p = -2x/R \tag{2.5}$$

As mentioned in the introduction, a finite-amplitude analysis for the growth of a single two-dimensional disturbance has been given elsewhere[3,4]. Here we wish to follow these two papers in considering the temporal growth of disturbances, but we wish to extend that work to consider the interaction of a two-dimensional disturbance with a three-dimensional one. The former disturbance has a wave-length $2\pi/\alpha$ (wave number α) in the x-direction while the latter is assumed to have the same wave number, α, in the x-direction, with a wave number β in the y-direction. We also consider the waves to propagate in the x-direction, but to have zero wave velocity in the y-direction; the three-dimensional disturbance is therefore a standing wave as far as the y-direction is concerned. (It is possible with a similar analysis to consider the case of a three-dimensional wave which does have a wave velocity in the y-direction; however we shall not pursue this point, here, but some relevant differences will be noted later. It is also possible to consider two disturbances whose x-wise wave numbers are different.)

We assume the velocity components u, v, w to be Fourier analysable in the form

$$u=\bar{u}+u_{10}\,e^{i\alpha x}+\tilde{u}_{10}e^{-i\alpha x}+(u_{11}\,e^{i\alpha x}+\tilde{u}_{11}\,e^{-i\alpha x})\cos\beta y+u_{20}\,e^{2i\alpha x}+\tilde{u}_{20}e^{-2i\alpha x}$$
$$+(u_{22}e^{2i\alpha x}+\tilde{u}_{22}e^{-2i\alpha x})\cos 2\beta y+u_{02}\cos 2\beta y+(u_{21}\,e^{2i\alpha x}+\tilde{u}_{21}e^{-2i\alpha x})\cos\beta y$$
$$+u_{01}\cos\beta y+.... \tag{2.6}$$

$$v=(v_{11}\,e^{i\alpha x}+\tilde{v}_{11}\,e^{-i\alpha x})\sin\beta y+(v_{22}\,e^{2i\alpha x}+\tilde{v}_{22}\,e^{-2i\alpha x})\sin 2\beta y+v_{02}\sin 2\beta y$$
$$+(v_{21}\,e^{2i\alpha x}+\tilde{v}_{21}\,e^{-2i\alpha x})\sin\beta y+v_{01}\sin\beta y+... \tag{2.7}$$

$$w=w_{10}e^{i\alpha x}+\tilde{w}_{10}e^{-i\alpha x}+(w_{11}e^{i\alpha x}+\tilde{w}_{11}e^{-i\alpha x})\cos\beta y+(w_{20}e^{2i\alpha x}+\tilde{w}_{20}e^{-2i\alpha x})$$
$$+(w_{22}e^{2i\alpha x}+\tilde{w}_{22}e^{-2i\alpha x})\cos 2\beta y+w_{02}\cos 2\beta y+(w_{21}e^{2i\alpha x}+\tilde{w}_{21}e^{-2i\alpha x})\cos\beta y$$
$$+w_{01}\cos\beta y+... \tag{2.8}$$

$$p=x\bar{p}(t)+\bar{\bar{p}}+(p_{10}e^{i\alpha x}+\tilde{p}_{10}e^{-i\alpha x})+(p_{11}e^{i\alpha x}+\tilde{p}_{11}e^{-i\alpha x})\cos\beta y+p_{20}e^{2i\alpha x}$$
$$+\tilde{p}_{20}e^{-2i\alpha x}+(p_{22}e^{2i\alpha x}+\tilde{p}_{22}e^{-2i\alpha x})\cos 2\beta y+p_{02}\cos 2\beta y$$
$$+(p_{21}e^{2i\alpha x}+\tilde{p}_{21}e^{-2i\alpha x})\cos\beta y+p_{01}\cos\beta y \tag{2.9}$$

In these expressions the tilde ($\sim$) denotes a complex conjugate, the first suffix refers to the harmonic of the streamwise wave number α, and the second suffix refers to the harmonic of the spanwise wave number β. The function $\tilde{u}$ is the mean velocity in the x-direction, the average being taken with respect to x and y. All the u, v, w and p quantities except $\tilde{p}(t)$ are function of both z and t. (In the case of a wave propagating in the y-direction, there would be a mean velocity, $\tilde{v}$, in this direction, generated by a Reynolds stress).

It can be seen that the two-dimensional basic disturbances is represented by terms of the form $e^{i\alpha x}$ and $e^{-i\alpha x}$, while the three-dimensional basic disturbance is represented by terms of the form $e^{i\alpha x}\cos\beta y$ and $e^{-i\alpha x}$ $\cos\beta y$. The other terms specifically included in formulae (2.6) to (2.9) arise from direct quadratic interaction between the different components of these basic disturbances. Higher harmonics will be discussed in Section 4, when we shall find that they do not affect the basic non-linear problem. Of the terms in the expression for the pressure, $\bar{p}(t)$ represents the imposed longitudinal pressure gradient, while $\bar{\bar{p}}(z, t)$ represents the pressure variation in the z-direction due to the effects of Reynolds stresses.

If we substitute (2.6) to (2.9) in equations (2.1) to (2.4), separate out similar harmonic components, and eliminate the pressure terms in all except the mean equations, we obtain the following partial differential equations, where it should be noted that the prime and the operator D are used interchangeably to denote differentiation with respect to z. It is convenient first to define the operators

$$L(\alpha,\beta) \equiv \left(\bar{u}-\frac{i}{\alpha}\frac{\partial}{\partial t}\right)(D^2-\alpha^2-\beta^2)-\bar{u}''+\frac{i}{\alpha R}(D^2-\alpha^2-\beta^2)^2 \quad (2.10)$$

$$M(\alpha,\beta) \equiv \bar{u}-\frac{i}{\alpha}\frac{\partial}{\partial t}+\frac{i}{\alpha R}(D^2-\alpha^2-\beta^2) \quad (2.11)$$

The symbols U, V, W (with subscripts and bars) are described later, and then we have

(i) Mean terms

$$\frac{\partial \bar{u}}{\partial t}+\bar{U} = -\bar{p}+R^{-1}\bar{u}'' \qquad \overline{W} = -\bar{\bar{p}}_1 \quad (2.12)$$

(ii) $e^{i\alpha x}$ terms

$$L(\alpha,0)w_{10} = U'_{10}-i\alpha W_{10} \qquad u_{10} = \frac{i}{\alpha}w'_{10} \quad (2.13)$$

(iii) $e^{i\alpha x}\cos\beta y$, $e^{i\alpha x}\sin\beta y$ terms

$$L(\alpha,\beta)\,w_{11} = U'_{11}-\frac{i\beta}{\alpha}V'_{11}-\frac{i}{\alpha}(\alpha^2+\beta^2)W_{11} \quad (2.14)$$

$$M(\alpha,\beta)(\beta u_{11}+i\alpha v_{11}) = \frac{i\beta}{\alpha}w_{11}\bar{u}'+\frac{i\beta}{\alpha}U_{11}-V_{11} \quad (2.15)$$

$$i\alpha u_{11}+\beta v_{11}+w'_{11} = 0 \quad (2.16)$$

(iv) $e^{2i\alpha x}$ terms

$$L(2\alpha,0)\,w_{20} = U'_{20}-2i\alpha W_{20} \quad (2.17)$$

$$u_{20} = \frac{i}{2\alpha}w'_{20} \quad (2.18)$$

(v) $e^{2i\alpha x}\cos 2\beta y$, $e^{2i\alpha x}\sin 2\beta y$ terms

$$L(2\alpha,2\beta)\,w_{22} = U'_{22}-\frac{i\beta}{\alpha}V'_{22}-\frac{2i}{\alpha}(\alpha^2+\beta^2)W_{22} \quad (2.19)$$

$$M(2\alpha,2\beta)(\beta u_{22}+i\alpha v_{22}) = \frac{i\beta}{2\alpha}w_{22}\bar{u}'+\frac{i\beta}{2\alpha}U_{22}-\tfrac{1}{2}V_{22} \quad (2.20)$$

$$i\alpha u_{22}+\beta v_{22}+\tfrac{1}{2}w'_{22} = 0 \quad (2.21)$$

(vi) $\cos 2\beta y$, $\sin 2\beta y$ terms

$$[i\alpha L(\alpha,2\beta)]_{\alpha=0}w_{02} = 2\beta V'_{02}+4\beta^2 W_{02} \quad (2.22)$$

$$[i\alpha M(\alpha,2\beta)]_{\alpha=0}u_{02} = -w_{02}\bar{u}'-U_{02} \quad (2.23)$$

$$2\beta v_{02}+w'_{02} = 0 \quad (2.24)$$

(vii) $e^{2i\alpha x}\cos\beta y$, $e^{2i\alpha x}\sin\beta y$ terms

$$L(2\alpha,\beta)w_{21}=U'_{21}-\frac{i\beta}{2\alpha}V'_{21}-\frac{i}{2\alpha}(4\alpha^2+\beta^2)W_{21} \tag{2.25}$$

$$M(2\alpha,\beta)(\beta u_{21}+2i\alpha v_{21})=\frac{i\beta}{2\alpha}w_{21}\bar{u}'+\frac{i\beta}{2\alpha}U_{21}-V_{21} \tag{2.26}$$

$$2i\alpha u_{21}+\beta v_{21}+w'_{21}=0 \tag{2.27}$$

(viii) $\cos\beta y$, $\sin\beta y$ terms

$$[i\alpha L(\alpha,\beta)]_{\alpha=0}w_{01}=\beta V'_{01}+\beta^2 W_{01} \tag{2.28}$$

$$[i\alpha M(\alpha,\beta)]_{\alpha=0}u_{01}=-w_{01}\bar{u}'-U_{01} \tag{2.29}$$

$$\beta v_{01}+w'_{01}=0 \tag{2.30}$$

In all these differential equations the symbols $\bar{U}$, $\bar{W}$, U_{mn}, V_{mn}, W_{mn} (m, n integral) represent the terms which arise from non-linear interactions. Representative ones are

$$\bar{U}\equiv w_{10}\tilde{u}'_{10}+\tilde{w}_{10}u'_{10}+\tfrac{1}{2}(w_{11}\tilde{u}'_{11}+\tilde{w}_{11}u'_{11}-\beta v_{11}\tilde{u}_{11}-\beta\tilde{v}_{11}u_{11})+\text{ higher-order terms} \tag{2.31}$$

$$U_{10}\equiv i\alpha\tilde{u}_{10}u_{20}+\tfrac{1}{2}i\alpha\tilde{u}_{11}u_{21}+\tfrac{1}{2}i\alpha u_{01}u_{11}-\tfrac{1}{2}\beta v_{11}u_{01}-\tfrac{1}{2}\beta\tilde{v}_{11}u_{21}-\tfrac{1}{2}\beta v_{21}\tilde{u}_{11} -\tfrac{1}{2}\beta v_{01}u_{11}+\tilde{w}_{10}u'_{10}+\tfrac{1}{2}w_{11}u'_{01}+\tfrac{1}{2}\tilde{w}_{11}u'_{21}+w_{20}\tilde{u}'_{10}+\tfrac{1}{2}w_{21}\tilde{u}'_{11} +\tfrac{1}{2}w_{01}u'_{11}+\text{higher-order terms} \tag{2.32}$$

$$U_{20}\equiv i\alpha u_{10}^2+\tfrac{1}{2}i\alpha u_{11}^2-\tfrac{1}{2}\beta v_{11}u_{11}+w_{10}u'_{10}+\tfrac{1}{2}w_{11}u'_{11}+\text{higher-order terms} \tag{2.33}$$

The higher-order terms will be discussed in Section 4, and will be shown to be negligible to the order of magnitude we wish to consider.

The functions $\bar{U}$ and $\bar{W}$ are the well-known Reynolds stresses; the first equation (2.12) shows that the mean velocity, $\tilde{u}$, is dependent on $\bar{U}$, while the second equation (2.12) gives the pressure gradient required to balance the Reynolds stress in the z-direction. Each of the (viii) sets of equations given above is dependent on the others through the non-linear interaction functions (U_{mn} etc.), which represent the interdependence of the many harmonic components of oscillation. Thus the eight sets of equations must be solved jointly.

The velocity components v_{10} and v_{20} are identically zero, as are the corresponding interaction functions V_{10} and V_{20}, because the flow is a standing wave in the y-direction. If the flow were to possess a wave velocity in the y-direction, the velocities v_{10} and v_{20} would be non-zero.

Before concluding this section we give a preliminary discussion of the physical meaning to be attached to each of the eight groups of equations given above. Group (i) represents the distorted mean motion in the

x-direction, averaged with respect to the directions x and y. Groups (vi) and (viii) represent other harmonic components which do not vary with x, but whose velocities are periodic in y. These are generated as a harmonic of the three-dimensional wave alone, and by the interaction between the two waves, respectively. Groups (ii) and (iii) represent, respectively, the two-dimensional and three-dimensional basic disturbances, each being modified (from the form of linearized theory) by non-linear interactions. Groups (iv), (v) and (vii) represent harmonic of the basic oscillatory disturbances, arising from the two-dimensional oscillation (iv), from the three-dimensional oscillation (v), and from the interaction of the oscillations (vii). From the experimental point of view the parts (i) (vi) and (viii), which are not periodic in x, are particularly significant.

3. METHOD OF SOLUTION

Watson[4] has studied the problem posed by differential equations analogous to those of the previous section, for the case of a single two-dimensional disturbances, and has shown how the solution may be expanded in a certain type of series. Here we wish to generalize Watson's work to the case of two interacting disturbances, but we shall study only terms up to third order in amplitude in the expansion instead of the whole series. (This amounts to considering disturbances whose amplification rates are sufficiently small.).

An essentially non-linear feature shown by the work of Stuart[3] and Watson[4] is the divergence, at finite amplitudes, from an exponential rate of growth. On this point, the present work differs from that of Lin[1] and Benney[2], where the rate of growth is always exponential with time (except for the special case when the time rates of growth of the basic sturbances — $e^{i\alpha x}$ and $e^{i\alpha x}\cos\beta y$—are identically zero).

We look for a solution of equations (2.12) to (2.30) in the form

$$\bar{u} = \bar{u}_l + |A|^2 f_1^{(1)} + |B|^2 f_1^{(2)} + \dots \tag{3.1}$$

where $\bar{u}_l \equiv 1-z^2$ is the laminar undisturbed flow, together with

$$w_{10} = A(\psi_{10} + |A|^2\psi_{10}^{(11)} + |B|^2\psi_{10}^{(12)} + \dots) + \tilde{A}B^2\psi_{10}^{(13)} + \dots \tag{3.2}$$

$$u_{10} = A(\phi_{10} + |A|^2\phi_{10}^{(11)} + |B|^2\phi_{10}^{(12)} + \dots) + \tilde{A}B^2\phi_{10}^{(13)} + \dots \tag{3.3}$$

$$w_{11} = B(\psi_{11} + |A|^2\psi_{11}^{(11)} + |B|^2\psi_{11}^{(12)} \dots) + \tilde{B}A^2\psi_{11}^{(13)} + \dots \tag{3.4}$$

$$u_{11} = B(\phi_{11} + |A|^2\phi_{11}^{(11)} + |B|^2\phi_{11}^{(12)} + \dots) + \tilde{B}A^2\phi_{11}^{(13)} + \dots \tag{3.5}$$

$$v_{11} = B(\chi_{11} + |A|^2\chi_{11}^{(11)} + |B|^2\chi_{11}^{(12)} + \dots) + \tilde{B}A^2\chi_{11}^{(13)} + \dots \tag{3.6}$$

$$u_{20} = A^2\psi_{20}^{(01)} + B^2\psi_{20}^{(02)} + \dots \tag{3.7}$$

$$w_{20} = A^2\phi_{20}^{(01)} + B_2\phi_{20}^{(02)} + \dots \tag{3.8}$$

$$w_{22} = B^2\psi_{22} + \dots \tag{3.9}$$

$$u_{22} = B^2\phi_{22} + \dots \tag{3.10}$$

$$v_{22} = B^2\chi_{22} + \dots \tag{3.11}$$

$$w_{02} = |B|^2\psi_{02} + \dots \tag{3.12}$$

$$u_{02} = |B|^2\phi_{02} + \dots \tag{3.13}$$

$$v_{02} = |B|^2\chi_{02} + \dots \tag{3.14}$$

$$w_{21} = AB\psi_{21} + \dots \tag{3.15}$$

$$u_{21} = AB\phi_{21} + \dots \tag{3.16}$$

$$v_{21} = AB\chi_{21} + \dots \tag{3.17}$$

$$w_{01} = A\tilde{B}\psi_{01} + \tilde{A}B\tilde{\psi}_{01} + \cdots \tag{3.18}$$

$$u_{01} = A\tilde{B}\phi_{01} + \tilde{A}B\tilde{\phi}_{01} + \cdots \tag{3.19}$$

$$v_{01} = A\tilde{B}\chi_{01} + \tilde{A}B\tilde{\chi}_{01} + \cdots \tag{3.20}$$

It will be seen that the two fundamentals are given to third order in amplitude, and the other components only to second order. The functions $A(t)$ and $B(t)$ are assumed to be given by the relations

$$\frac{dA}{dt} = A(a_0 + a_1^{(1)}|A|^2 + a_1^{(2)}|B|^2 + \cdots) + a_1^{(3)}\tilde{A}B^2 + \cdots \tag{3.21}$$

$$\frac{dB}{dt} = B(b_0 + b_1^{(1)}|A|^2 + b_1^{(2)}|B|^2 + \cdots) + b_1^{(3)}\tilde{B}A^2 + \cdots \tag{3.22}$$

which render (3.1)—(3.20) consistent with (2.12)—(2.30).
In formulae (3.1) to (3.20) the functions f, ψ, ϕ and χ are dependent on z alone. On the other hand A and B are functions of t alone. The constants $a_1^{(1)}$, $a_1^{(2)}$, $a_1^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_1^{(3)}$ are constants to be determined. The constants a_0 and b_0 arise from linearized theory and may be written

$$a_0 = -i\alpha c_1, \quad b_0 = -i\alpha c_2 \tag{3.23}$$

(If the non-linear terms in (3.21) and (3.22) are ignored, A and B are proportional to exp($-i\alpha c_1 t$) and exp($-i\alpha c_2 t$) respectively, the time-dependent behaviour of linearized theory).

We must now consider the dependence of the various interaction functions U, V, W on A and B; in this discussion it is convenient to refer to functions by the double suffixes; for examples "21" refers to the coefficient of $e^{2i\alpha x}\cos\beta y$, the first suffix referring to the wave number in the x-direction, and the second suffix to the wave number in the y-direction. It appears that the mean functions ($\bar{U}$, $\bar{W}$) contain terms proportional to $|A|^2$ and to $|B|^2$; the 10-components contain terms proportional to $A|A|^2$,

$A|B|^2$ and $\tilde{A}B^2$. These and other properties are shown in the table immediately below.

TABLE 1

Components U, V, W	Time-dependence
$\overline{U}, \overline{W}$	$\|A\|^2, \|B\|^2$
10	$A\|A\|^2, A\|B\|^2, \tilde{A}B^2$
11	$B\|A\|^2, B\|B\|^2, \tilde{B}A^2$
20	A^2, B^2
22	B^2
02	$\|B\|^2$
21	AB
01	$A\tilde{B}, \tilde{A}B$

The U, V, W functions for the mean ($\overline{U}$, $\overline{W}$), 02 and 01 cases are real; in the 01 case the (z-dependent) coefficients of $A\tilde{B}$ and $\tilde{A}B$ are complex conjugates. The A, B dependencies given in the above table refer, of course, only to the most important terms, higher powers of A and B not being given explicity; however the terms given are the lowest-order interaction terms which need to be considered.

We now write

$$\overline{U} = |A|^2\overline{U}^{(1)} + |B|^2\overline{U}^{(2)} \tag{3.24}$$

$$U_{10} = A|A|^2U_{10}^{(1)} + A|B|^2U_{10}^{(2)} + \tilde{A}B^2U_{10}^{(3)} \tag{3.25}$$

$$U_{11} = B|A|^2U_{11}^{(1)} + B|B|^2U_{11}^{(2)} + \tilde{B}A^2U_{11}^{(3)} \tag{3.26}$$

$$U_{20} = A^2U_{20}^{(1)} + B^2U_{20}^{(2)} \tag{3.27}$$

$$U_{22} = B^2U_{22}^{(0)} \tag{3.28}$$

$$U_{02} = |B|^2U_{02}^{(0)} \tag{3.29}$$

$$U_{21} = ABU_{21}^{(0)} \tag{3.30}$$

$$U_{01} = A\tilde{B}U_{01}^{(0)} + \tilde{A}B\tilde{U}_{01}^{(0)} \tag{3.31}$$

with similar definitions for the V and W functions. The functions $U_{mn}^{(r)}$ depend only an z.

The set of partial differential equations (2.12) to (2.30) can now be reduced to a set of ordinary differential equations by separating coefficients of A, B, $|A|^2$, etc. Firstly we define

$$L(\alpha,\beta,c) \equiv (\bar{u}_l - c)(D^2-\alpha^2-\beta^2) - \bar{u}_l'' + \frac{i}{\alpha R}(D^2-\alpha^2-\beta^2)^2 \tag{3.32}$$

$$M(\alpha,\beta,c) \equiv (\bar{u}_l - c) + \frac{i}{\alpha R}(D^2-\alpha^2-\beta^2) \tag{3.33}$$

$$N(\beta,c) \equiv -i\alpha c - \frac{1}{R}(D^2-\beta^2) \tag{3.34}$$

Substituting (3.1) to (3.23) into (2.12) to (2.30) using (3.24) to (3.31), and separating as described above, we obtain the following sets of ordinary differential equations, where the orders of A and B are indicated on the left-hand side:

(i) Mean terms

$$0 = -\bar{p}^{(0)} + R^{-1}\bar{u}_e'' \tag{3.35}$$

$$|A|^2:\ 2\alpha c_{1i} f_1^{(1)} + \bar{U}^{(1)} = -\bar{p}^{(11)} + R^{-1} f_1^{(1)''} \tag{3.36}$$

$$|B|^2:\ 2\alpha c_{2i} f_i^{(2)} + \bar{U}^{(2)} = -\bar{p}^{(12)} + R^{-1} f_1^{(2)''} \tag{3.37}$$

where $\bar{p}$ is assumed to be expandable in the form

$$\bar{p} = \bar{p}^{(0)} + \bar{p}^{(11)}|A|^2 + \bar{p}^{(12)}|B|^2 + \ldots \tag{3.38}$$

It is necessary to specify an overall condition on the mean motion, such as constant pressure gradient or constant mass flux; the former condition yields $\bar{p}^{(11)} = \bar{p}^{(12)} = 0$ and the latter condition yields appropriate values for $\bar{p}^{(11)}$ and $\bar{p}^{(12)}$, where $\bar{p}^{(0)}$ is specified by the basic laminar flow.

(ii) $e^{i\alpha x}$ terms

$$A:\begin{cases} L(\alpha, 0, c_1)\psi_{10} = 0 & (3.39) \\ i\alpha\varphi_{10} + \psi_{10}' = 0 & (3.40) \end{cases}$$

$$A|A|^2:\begin{cases} L(\alpha, 0, c_1 + 2ic_{1i})\psi_{10}^{(11)} = U_{10}^{(1)'} - i\alpha W_{10}^{(1)} + \left(\dfrac{ia_1^{(1)}}{\alpha} - f_1^{(1)}\right)(D^2 - \alpha^2)\psi_{10} \\ \qquad + f_1^{(1)''}\psi_{10} & (3.41) \\ i\alpha\phi_{10}^{(11)} + \psi_{10}^{(11)'} = 0 & (3.42) \end{cases}$$

$$A|B|^2:\begin{cases} L(\alpha, 0, c_1 + 2ic_{2i})\psi_{10}^{(12)} = U_{10}^{(2)'} - i\alpha W_{10}^{(2)} + \left(\dfrac{ia_1^{(2)}}{\alpha} - f_1^{(2)}\right)(D^2 - \alpha^2)\psi_{10} \\ \qquad + f_1^{(2)''}\psi_{10} & (3.43) \\ i\alpha\phi_{10}^{(12)} + \psi_{10}^{(12)'} = 0 & (3.44) \end{cases}$$

$$\tilde{A}B^2:\begin{cases} L(\alpha, 0, c_1 - 2c_{1r} + 2c_2)\psi_{10}^{(13)} = U_{10}^{(3)'} - i\alpha W_{10}^{(3)} + \dfrac{ia_1^{(3)}}{\alpha}(D^2 - \alpha^2)\psi_{10} & (3.45) \\ i\alpha\phi_{10}^{(13)} + \psi_{10}^{(13)'} = 0 & (3.46) \end{cases}$$

(iii) $e^{i\alpha x}\cos\beta y$, $e^{i\alpha x}\sin\beta y$ terms

$$B:\begin{cases} L(\alpha, \beta, c_2)\psi_{11} = 0 & (3.47) \\ M(\alpha, \beta, c_2)(\beta\phi_{11} + i\alpha\chi_{11}) = \dfrac{i\beta}{\alpha}\psi_{11}\bar{u}_l' & (3.48) \\ i\alpha\phi_{11} + \beta\chi_{11} + \psi_{11}' = 0 & (3.49) \end{cases}$$

$$B|A|^2: \left\{ \begin{aligned} & L(\alpha, \beta, c_2+2ic_{1i})\psi_{11}^{(11)} = U_{11}^{(1)\prime} - \frac{i\beta}{\alpha} V_{11}^{(1)\prime} - \frac{i}{\alpha}(\alpha^2+\beta^2) W_{11}^{(1)} \\ & \qquad + \left(\frac{ib_1^{(1)}}{\alpha} - f_1^{(1)}\right)(D_2-\alpha_2-\beta_2)\psi_{11} + f_1^{(1)}\psi_{11} \qquad (3.50) \\ & M(\alpha, \beta, c_2+2ic_{1i})(\beta\phi_{11}^{(11)}+i\alpha\chi_{11}^{(11)}) = \frac{i\beta}{\alpha} U_{11}^{(1)} - V_{11}^{(1)} + \frac{i\beta}{\alpha}\psi_{11}^{(11)}\bar{u}_l' \\ & \qquad + \left(\frac{ib_1^{(1)}}{\alpha} - f_{1i}^{(1)}\right)(B\phi_{11}+i\alpha\chi_{11}) \\ & \qquad + \frac{i\beta}{\alpha}\psi_{11} f_1^{(1)\prime} \qquad (3.51) \\ & i\alpha\phi_{11}^{(11)} + \beta\chi_{11}^{(11)} + \psi_{11}^{(11)\prime} = 0 \qquad (3.52) \end{aligned} \right.$$

$$B|B|^2: \left\{ \begin{aligned} & L(\alpha, \beta, c_2+2ic_{2i})\psi_{11}^{(12)} = U_{11}^{(2)\prime} - \frac{i\beta}{\alpha} V_{11}^{(2)\prime} - \frac{i}{\alpha}(\alpha^2+\beta^2) W_{11}^{(2)} \\ & \qquad + \left(\frac{ib_1^{(2)}}{\alpha} - f_1^{(2)}\right)(D^2-\alpha^2-\beta^2)\psi_{11} \\ & \qquad + f_1^{(2)\prime\prime}\psi_{11} \qquad (3.53) \\ & M(\alpha, \beta, c_2+2ic_{2i})(\beta\phi_{11}^{(12)}+i\alpha\chi_{11}^{(12)}) = \frac{i\beta}{\alpha}\psi_{11}^{(12)}\bar{u}_l' + \frac{i\beta}{\alpha} U_{11}^{(2)} - V_{11}^{(2)} \\ & \qquad + \left(\frac{ib_1^{(2)}}{\alpha} - f_1^{(2)}\right)(\beta\phi_{11}+i\alpha\chi_{11}) \\ & \qquad + \frac{i\beta}{\alpha}\psi_{11} f_1^{(2)\prime} \qquad (3.54) \\ & i\alpha\phi_{11}^{(12)} + \beta\chi_{11}^{(12)} + \psi_{11}^{(12)\prime} = 0 \qquad (3.55) \end{aligned} \right.$$

$$\tilde{B}A^2: \left\{ \begin{aligned} & L(\alpha, \beta, c_2+2c_{2r}+2c_1)\psi_{11}^{(13)} = U_{11}^{(3)\prime} - \frac{i\beta}{\alpha} V_{11}^{(3)\prime} - \frac{i}{\alpha}(\alpha^2+\beta^2) W_{11}^{(3)} \\ & \qquad + \frac{ib_1^{(3)}}{\alpha}(D^2-\alpha^2-\beta^2)\psi_{11} \qquad (3.56) \\ & M(\alpha, \beta, c_2-2c_{2r}+2c_1)(\beta\phi_{11}^{(13)}+i\alpha\chi_{11}^{(13)}) = \frac{i\beta}{\alpha}\psi_{11}^{(13)}\bar{u}_l' + \frac{i\beta}{\alpha} U_{11}^{(3)} - V_{11}^{(3)} \\ & \qquad + \frac{ib_1^{(3)}}{\alpha}(\beta\phi_{11}+i\alpha\chi_{11}) \\ & \qquad (3.57) \\ & i\alpha\phi_{11}^{(13)} + \beta\chi_{11}^{(13)} + \psi_{11}^{(13)\prime} = 0 \qquad (3.58) \end{aligned} \right.$$

(iv) $e^{2i\alpha x}$ terms

$$A^2:\quad \begin{cases} L(2\alpha, 0, c_1)\psi_{20}^{(01)} = U_{20}^{(1)\prime} - 2i\alpha W_{20}^{(1)} & (3.59)\\ 2i\alpha\phi_{20}^{(01)} + \psi_{20}^{(01)\prime} = 0 & (3.60)\end{cases}$$

$$B^2:\quad \begin{cases} L(2\alpha, 0, c_2)\psi_{20}^{(02)} = U_{20}^{(2)\prime} - 2i\alpha W_{20}^{(2)} & (3.61)\\ 2i\alpha\phi_{20}^{(02)} + \psi_{20}^{(02)\prime} = 0 & (3.62)\end{cases}$$

(v) $e^{2i\alpha x}\cos 2\beta y$, $e^{2i\alpha x}\sin 2\beta y$ terms

$$B^2:\quad \begin{cases} L(2\alpha, 2\beta, c_2)\psi_{22} = U_{22}^{(0)\prime} - \dfrac{i\beta}{\alpha}V_{22}^{(0)\prime} - \dfrac{2i}{\alpha}(\alpha^2+\beta^2)W_{22}^{(0)} & (3.63)\\[2ex] M(2\alpha, 2\beta, c_2)(\beta\phi_{22}+i\alpha\chi_{22}) = \dfrac{i\beta}{2\alpha}\psi_{22}\bar{u}_l' + \dfrac{i\beta}{2\alpha}U_{22}^{(0)} - \tfrac{1}{2}V_{22}^{(0)} & (3.64)\\[2ex] i\alpha\phi_{22} + \beta\chi_{22} + \tfrac{1}{2}\psi_{22}' = 0 & (3.65)\end{cases}$$

(vi) $\cos 2\beta y$, $\sin 2\beta y$ terms

$$|B|^2:\quad \begin{cases} (D^2-4\beta^2)N(2\beta, 2ic_{2i})\psi_{02} = 2\beta V_{02}^{(0)\prime} + 4\beta^2 W_{02}^{(0)} & (3.66)\\ N(2\beta, 2ic_{2i})\phi_{02} = -U_{02}^{(0)} - \psi_{02}\bar{u}_1' & (3.67)\\ 2\beta\chi_{02} + \psi_{02}' = 0 & (3.68)\end{cases}$$

(vii) $e^{2i\alpha x}\cos\beta y$, $e^{2i\alpha x}\sin\beta y$ terms

$$AB:\quad \begin{cases} L(2\alpha, \beta, \tfrac{1}{2}(c_1+c_2))\psi_{21} = U_{21}^{(0)\prime} - \dfrac{i\beta}{2\alpha}V_{21}^{(0)\prime} - \dfrac{i}{2\alpha}(4\alpha^2+\beta^2)W_{21}^{(0)} & (3.69)\\[2ex] M(2\alpha, \beta, \tfrac{1}{2}(c_1+c_2)\chi\beta\phi_{21} + i\alpha\chi_{21}) = \dfrac{i\beta}{2\alpha}(\psi_{21}\bar{u}_l' + U_{21}^{(0)}) - V_{21}^{(0)} & (3.70)\\[2ex] 2i\alpha\phi_{21} + \beta\chi_{21} + \psi_{21}' = 0 & (3.71)\end{cases}$$

(viii) $\cos\beta y$, $\sin\beta y$ terms

$$AB:\quad \begin{cases} (D^2-\beta^2)N(\beta, c_1-\tilde{c}_2)\psi_{01} = \beta V_{01}^{(0)\prime} + \beta^2 W_{01}^{(0)} & (3.72)\\ N(\beta, c_1-\tilde{c}_2)\phi_{01} = -U_{01}^{(0)} - \psi_{01}\bar{u}_l' & (3.73)\\ \beta\chi_{01} + \psi_{01}' = 0 & (3.74)\end{cases}$$

The boundary conditions which we wish to apply on the differential equations described above are that the velocity components shall vanish at $z = \pm 1$; and that the two fundamentals shall be antisymmetrical about $z = 0$. It then follows that w_{10} and w_{11} will be even functions of z, and u_{10}, u_{11} and v_{11} odd functions of z. Further considerations show that $\bar{u}$, u_{20}, u_{22}, v_{22}, u_{02}, v_{02}, u_{21}, v_{21}, u_{01} and v_{01} are to be even, and w_{20}, w_{22}, w_{02}, w_{21}, w_{01} odd functions of z. Thus the boundary conditions are

$$\text{at}\quad z = 1: \bar{u}_1 = f_1^{(1)} = f_1^{(2)} = \psi = \psi' = \phi = \chi = 0 \tag{3.75}$$

$$\text{at}\quad \left.\begin{aligned} z = 0: \psi_{1n}' = \psi_{1n}'' = \phi_{2n}' = \chi_{2n}' = \phi_{0n}' = \chi_{0n}' = 0\\ \psi_{2n} = \psi_{2n}'' = \psi_{0n} = \psi_{0n}'' = \phi_{1n} = \chi_{1n} = 0\end{aligned}\right\} \tag{3.76}$$

In (3.75) ψ, ϕ, χ denote any function occurring in equations (3.39) to (3.74), and in (3.76) ψ_{1n}, ψ_{2n}, etc. also refer to any the functions in (3.39) to (3.74).

For the purposes of the present paper it is unnecessary to quote the precise algebraic forms of the interaction function $U_{10}^{(1)}$, $U_{20}^{(1)}$, etc. It suffices to say that the interaction functions which arise in the sets of equations (i) and (iv) to (viii) depend only on the functions ψ_{10}, ϕ_{10}, ψ_{11}, ϕ_{11}, χ_{11} associated with the two fundamental Tollmien-Schlichting wave disturbances. On the other hand the interaction functions arising in equations (3.41) to (3.46) and (3.50) to (3.58)—the equations for the third-order modifications of the fundamental waves—involve ψ_{10}, ϕ_{10}, ψ_{11}, ϕ_{11}, χ_{11} together with the solutions to equations (3.59) to (3.74). With the above knowledge it is possible to describe the sequence in which calculations can be done.

Procedure of Calculation

(a) Evaluate the properties of the fundamental waves for given values of α, β and R (equations (3.39), (3.40), (3.47)–(3.49)). This leads to the determination of c_1, c_2, ψ_{10}, ϕ_{10}, ψ_{11}, ϕ_{11} and χ_{11} (see, for example, Lin's book[5] for a description of linearized theory).

(b) Solve equations (3.59) to (3.74), with appropriate boundary conditions, for the second-order (harmonic) components generated by interaction and self-interaction of the two fundamentals; it is assumed that, for example in (3.69), $2\alpha, \beta, \frac{1}{2}(c_1+c_2)$, R do not form eigenvalues, with similar assumptions in the other equations. Benney's paper[2] is concerned with the evaluation of functions of this kind (for a mixing-region velocity profile at large Reynolds number).

(c) Solve equation (3.41) for $\psi_{10}^{(11)}$ and $a_1^{(1)}$, equation (3.43) for $\psi_{10}^{(12)}$ and $a_1^{(2)}$, equation (3.45) for $\psi_{10}^{(13)}$ and $a_1^{(3)}$, equation (3.50) for $\psi_{11}^{(11)}$ and $b_1^{(1)}$, equation (3.53) for $\psi_{11}^{(12)}$ and $b_1^{(2)}$ and equation (3.56) for $\psi_{11}^{(13)}$ and $b_1^{(3)}$. Succeding equations to those just nominated may then be used to calculate the ϕ and χ functions associated with the ψ's. References 3 and 4 are concerned with the formulation of the two-dimensional problem which, in the present notation, is the calculation of $\psi_{10}^{(11)}$ and $a_1^{(1)}$. A word of explanation about the mathematics involved in these calculations, especially with reference to the values of $a_1^{(1)}$, $a_1^{(2)}$, $a_1^{(3)}$, $b_1^{(1)}$, $b_1^{(2)}$, $b_1^{(3)}$, is perhaps appropriate here.

A typical case is afforded by equation (3.41)

$$L(\alpha, 0, c_1+2ic_{1i})\psi = g+\frac{ia_1^{(1)}}{\alpha}g_1 \tag{3.77}$$

where ψ represents $\psi_{11}^{(11)}$ and $g(z)$ and $g_1(z)$ are those parts of the right-hand side of (3.41) which are respectively independent of, and proportional to, $a_1^{(1)}$. The boundary conditions are, from (3.75) and (3.76)

$$\psi = \psi' = 0 \quad \text{at} \quad z = 1; \qquad \psi' = \psi''' = 0 \quad \text{at} \quad z = 0 \tag{3.78}$$

It can be shown that the essential feature of this problem is that the left-hand side of (3.77) differs only by a term of order c_{1i} from equation (3.39), while the boundary conditions are the same. Watson[4] has shown that, consequently, the dominant part of the solution of (3.77) is proportional to c_{1i}^{-1}. In spite of this we wish the solution to be regular in c_{1i} as c_{1i} tends to zero (as the neutral curve of Fig. 2 is approached), and Watson[4] and Stuart[3] have shown that, in order to ensure this, $a_1^{(1)}$ must take a definite value as $c_{1i} \to 0$.

In order to calculate this value we need to define the solution, Ψ, which satisfies the equation adjoint to (3.39),

$$L(\alpha, 0, c_1)\Psi \equiv (\bar{u}_l - c_1)(D^2 - \alpha^2)\Psi + 2\bar{u}_l'\Psi' + \frac{i}{\alpha R}(D^2 - \alpha^2)^2\Psi = 0 \tag{3.79}$$

subject to the adjoint boundary conditions,

$$\Psi = \Psi' = 0 \quad \text{at} \quad z = 1; \qquad \Psi' = \Psi''' = 0 \quad \text{at} \quad z = 0 \tag{3.80}$$

(Incidentally (3.79) is the vorticity equation for the $\exp(i\alpha x)$ fundamental, but (3.80) are not appropriate boundary conditions on vorticity.) It may be checked that if $\psi_{10} = \psi_{10g}$ is a general solution of (3.39), $\Psi_g = (D^2 - \alpha^2)\psi_{10g}$ is a general solution of (3.79). It can be shown[4] that the value of $a_1^{(1)}$ may be obtained by multiplying (3.77) by Ψ, integrating by parts and omitting a term of order c_{1i}; then we have

$$\frac{i a_1^{(1)}}{\alpha} = -\int_0^1 \Psi g \, \mathrm{d}z \Big/ \int_0^1 \Psi g_1 \, \mathrm{d}z \tag{3.81}$$

This procedure of calculation yields $a_1^{(1)}$ to order 1, the error being of order c_{1i}. (The case when the denominator of (3.81) vanishes requires special treatment—see ref. 4.)

The method described above for the calculation of $a_1^{(1)}$, equation (3.79 may be applied also to the calculation of $a_1^{(2)}$, $b_1^{(1)}$ and $b_1^{(2)}$ by ensuring that the solution of (3.43) is regular as c_{2i} tends to zero, that the solution of (3.50) is regular as c_{1i} tends to zero, and that the solution of (3.53) is regular as c_{2i} tends to zero. On the other hand, in the case of equation (3.45), the left-hand side does not differ by a small amount from the left-hand side of (3.39) because, for a finite value of the spanwise wave number (β), c_{1r} and c_{2r} may differ by as much as 15 per cent. Thus the left-hand side of (3.45) does not differ by a small amount from (3.39). Consequently $a_1^{(3)}$ may have any value and, in particular, we may choose it to be zero for even then the solution is regular. Similarly, from equation (3.56), we may choose $b_1^{(3)}$ to be zero. (It needs to be mentioned, however, that if we considered $\beta \to 0$ the above conclusion that $a_1^{(3)}$ and $b_1^{(3)}$ may be chosen to be zero identically would not be valid, for in that case $c_1 \to c_2$; an argument similar to that for $a_1^{(1)}$ would then yield finite values for $a_1^{(3)}$ and $b_1^{(3)}$.)

It will be seen shortly that c_{1i} and c_{2i} are required to be sufficiently small for the method of this paper to be valid. A glance at Figs. 3 and 4 of Watson's paper[8] shows that it is possible, for given α and R, to choose a finite value of β to lie within a range such that c_{2i} is small when c_{1i} is small. But, as mentioned above, the values of c_{1r} and c_{2r} may differ noticeably.

4. DISCUSSION

The conclusion that $a_1^{(3)} \equiv b_1^{(3)} \equiv 0$ for finite values of β (end of Section 3) means that the amplitude equations (3.21) and (3.22) reduce to

$$\frac{dA}{dt} = A(-i\alpha c_1 + a_1^{(1)}|A|^2 + a_1^{(2)}|B|^2 + \cdots) \tag{4.1}$$

$$\frac{dB}{dt} = B(-i\alpha c_2 + b_1^{(1)}|A|^2 + b_1^{(2)}|B|^2 + \cdots) \tag{4.2}$$

where a_0 and b_0 have the values (3.23). By multiplying (4.1) by $\tilde{A}$ and adding its complex conjugate, together with related operations on (4.2), we have

$$\frac{d|A|^2}{dt} = 2|A|^2(\alpha c_{1i} + a_{1r}^{(1)}|A|^2 + a_{1r}^{(2)}|B|^2 + \cdots) \tag{4.3}$$

$$\frac{d|B|^2}{dt} = 2|B|^2(\alpha c_{2i} + b_{1r}^{(1)}|A|^2 + b_{1r}^{(2)}|\,B\,||^2 + \cdots) \tag{4.4}$$

These equations define the squares of the moduli of the amplitudes A and B; it is possible to write down equations for $\tilde{A}/A$ and $B/\tilde{B}$, and these latter equations can be evaluated once $|A|^2$ and $|B|^2$ have been calculated (for a special case $B(t) = 0$, see refs. 3, 4).

The possible time-independent states of equilibrium of (4.3) and (4.4) are

$$\text{(i)}\quad |A|^2 = 0 \qquad |B|^2 = 0 \tag{4.5}$$

$$\text{(ii)}\quad |A|^2 = -\frac{\alpha c_{1i}}{a_{1r}^{(1)}} \qquad |B|^2 = 0 \tag{4.6}$$

$$\text{(iii)}\quad |A|^2 = 0 \qquad |B|^2 = -\frac{\alpha c_{21}}{b_{1r}^{(2)}} \tag{4.7}$$

$$\text{(iv)}\quad |A|^2 = \frac{\alpha(c_{1i}b_{1r}^{(2)} - c_{2i}a_{1r}^{(2)})}{b_{1r}^{(1)}a_{1r}^{(2)} - a_{1r}^{(1)}b_{1r}^{(2)}} \qquad |B|^2 = \frac{\alpha(c_{2i}a_{1r}^{(1)} - c_{1i}b_{1r}^{(1)})}{b_{1r}^{(1}a_{1r}^{(2)} - a_{1r}^{(1)}b_{1r}^{(2)}} \tag{4.8}$$

We shall assume that $a_{1r}^{(1)}$ and $b_{1r}^{(1)}$, as calculated by formula (3.81), have non-zero values as c_{1i} tends to zero, and that $a_{1r}^{(2)}$ and $b_{1r}^{(2)}$ have non-zero values as $c_{2i} \to 0$. It will be seen that $|A|^2$ and $|B|^2$ are of order $(ac_{1i}+bc_{2i})$, where a and b are constants, provided (in case (iv)) that the denominator of (4.8) is not small; each term in equations (4.3) and (4.4) is of order c_{1i}^2, $c_{1i}c_{2i}$ or c_{2i}^2. Higher-order terms (of order c_{1i}^3, etc.) have been neglected; they would arise if terms of higher order were included in equations (3.1) to (3.20), and if higher harmonics were considered. If desired, such effects may be calculated subsequently to the calculation described here, and will lead to the appropriate terms of order $|A|^6$, etc. in (4.3) and (4.4). This has been done by Watson[4] in the two-dimensional case. It should be noted that equations (4.3) and (4.4) are sufficient to determine $|A|^2$ and $|B|^2$ to order c_{1i} and c_{2i}, and that the calculation of $a_1^{(1)}$ etc., to order 1 is sufficiently accurate.

Any state of these four is meaningful only if both $|A|^2$ and $|B|^2$ are positive. Solution (i) is, of course, the basic laminar motion, which is unstable if either c_{1i} or c_{2i} is positive or if both are positive. Solution (ii) has been studied in part in refs. 3 and 4, where it is shown that the solution for $B(t) \equiv 0$ is

$$|A|^2 = \frac{ac_{1i}Ce^{2\alpha c_{1i}t}}{1-a_{1r}^{(1)}Ce^{2\alpha c_{1i}t}} \tag{4.9}$$

where C is an arbitrary real constant. If $a_{1r}^{(1)} < 0$ an equilibrium state exists for $c_{1i} > 0$, and then (4.9) yields (4.6) as $t \to +\infty$, and the result of linearized theory as $t \to -\infty$ (Fig. 3a). On the other hand $a_{1r}^{(1)} > 0$ yields an equilibrium for $c_{1i} < 0$, and then (4.9) yields (4.6) as $t \to -\infty$ (Fig. 3b). The importance of the latter result is that instability would only exist if the amplitude lay above a threshold value (which would be a state of unstable equilibrium); this would be in accordance with the work described in ref. 10. Calculations of $a_1^{(1)}$ have not yet been completed to determine which of the two types of behaviour (Fig. 3a, b) occurs in a given range of Reynolds number and wave number. Solution (iii) is similar to solution (ii) in type, but represents a three-dimensional oscillation.

Solution (iv) is more novel and represents (provided both of (4.8) are positive) an equilibrium state consisting of a combination of the two-dimensional oscillation with the three-dimensional one.

It is felt that an important feature of an analysis of the type described in this paper lies in the determination of the relative stability (see, for example, Stoker[13]) of the three possible equilibria, (4.6)–(4.8); if we have two competing disturbances, we may determine which has a tendency to dominate. It is to be emphasized that the term "relative stability" is used to mean stability within the framework of (4.3) and (4.4), but not with respect to all hydrodynamic disturbances. In the light of the experimental facts that three-dimensional oscillations are often prevalent, it is possible that (4.7) or (4.8), rather than (4.6), is the appropriate solution at many Reynolds numbers. Once the constants $a_1^{(1)}$, $a_1^{(2)}$, $b_1^{(1)}$, $b_1^{(2)}$ have been evaluated, a study of equations (4.3) and (4.4) will yield information as to whether this is so. A similar analysis could be done for two disturbances, which are both three-dimensional but have different wave numbers.

As in the case of the two-dimensional solution described in refs. 3 and 4, it seems likely that for convergence of the series in this paper c_{1i} and c_{2i} must be small compared with $(\alpha R)^{-1/3}$, to ensure uniform convergence in the region of the critical layer.

As mentioned in the introduction the analysis of Benney[2] is concerned mainly with the solution of equations (3.47) to (3.74), that is with the second-order effects, once the two fundamentals have been determined. (The solution given in (2) is valid at large Reynolds number in a mixing-region flow.) The *a* and *b* constants of equations (4.1) to (4.4) are assumed in (2) to be zero, and the disturbances grow exponentially with time. Consequently Benney's analysis is not concerned with the determination of the relative stability of the two fundamentals, in the sense of equations (4.3, 4.4). His paper is concerned mainly with the second-orders flows associated with equations (3.66) to (3.68) and (3.72) to (3.74). The cos $2\beta y$ terms (3.66–3.68) yield a steady flow with a spanwise wave-length of π/β; the cos βy terms (3.72–3.74) yield a flow with spanwise wave-length $2\pi/\beta$ and a frequency of $|\alpha(c_{1r}—c_{2r})|$, but the frequency is zero in Benney's work because of the simplifying assumption $c_1 = c_2$.

In the present writer's opinion it is preferable to include the oscillatory feature of the cos βy terms ($c_1 \neq c_2$) because, although its frequency is only at most 1/6th or 1/7th of the fundamental frequencies (in the Blasius case), a slow phase change of the cos βy term occurs relative to the exp $(i\alpha x)$ cos βy fundamental. Consequently the streamwise positions where the cos βy streamwise vorticity (which is proportional to $(D^2—\beta^2)\psi_{01}$ of (3.72)) reinforces the streamwise vorticity of the fundamental will vary in time in our assumed parallel flow (corresponding to a spatial variation in experiment). According to Gortler and Witting[14] streamwise vorticity will be developed at positions where the streamlines are concave relative to an observer moving with the wave speed, whereas in experiments of Klebanoff it occurs at positions where the stream lines are convex. With the assumption mentioned above, Benney's theory[2] agrees qualitatively with experiment, but the calculations were done on a mixing-region velocity profile. The position of reinforcement of streamwise vorticity can vary in the present theory; in fact, with the frequency difference quoted above, the position of vorticity reinforcement would move from a "concave" to "convex" position in only 3 or 4 wave-lengths. This feature seems to be one requiring further theoretical and experimental investigation.

The spanwise energy flux (per unit area in the xz-plane) which is involved in the flow may be calculated to be $\overline{Ev}$, where $E = \frac{1}{2}\varrho(u^2+v^2+w^2)$. Since u includes the basic mean motion, $\overline{Ev}$ may be approximated to first order by $\frac{1}{2}\varrho\overline{u^2v}$. It can be shown that

$$\overline{u^2v} = \bar{u}^2(v_{01}\sin\beta y+v_{02}\sin 2\beta y)+2\bar{u}[(u_{10}\tilde{v}_{11}+\tilde{u}_{10}v_{11})\sin\beta y + \tfrac{1}{2}(u_{11}\tilde{v}_{11}+\tilde{u}_{11}v_{11})\sin 2\beta y] \qquad (4.10)$$

to first order. The terms proportional to sin $2\beta y$ are proportional also to $|B|^2$, while those proportional to sin βy are proportional to $A\tilde{B}$ or $\tilde{A}B$ and therefore oscillate with a frequency $|\alpha(c_{1r}-c_{2r})|$. (In experiment this corresponds to a periodicity in x.) Furthermore the terms proportional to $\bar{u}^2$ represent a spanwise transfer of mean-flow energy, whereas the terms proportional to $\bar{u}$ represent a transfer of fluctuation energy. Spanwise energy transfer has been measured by Klebanoff and Tidstrom, but the experimental values are of order 10^2–10^3 times bigger than the values for which one might expect the present perturbation theory to be quantitatively valid.

This fact leads directly to the point that a perturbation theory of the type described here, as with the work of refs. 1–4, cannot be expected to describe quantitative features of the flow at any Reynolds number, wave number and frequency. Perturbation theories for non-linear instability have two applications: one is to show qualitatively some of the principal non-linear effects in the instability of actual flows, and it is felt that this can certainly be done for simple disturbances by analyses of the type of refs. 1–4, and the present paper, even without detailed numerical calculation. The second application is to the calculation of quantitative aspects of the flow at suitable Reynolds number, wave number and frequency, so that comparison with an adequately-controlled experiment becomes possible at the same Reynolds number, wave number and frequency. It is to be hoped that such a careful experiment can be done, for example, in plane Poiseuille flow to check the quantitative aspects. An attempt of this kind is in progress at the NPL under the direction of Mr. P. Bradshaw. (A difficulty of quantitative comparison in the Blasius case is the growth of the boundary layer, a feature which has not been incorporated in the theory.) In connexion with experimental and theoretical comparisons, it should be noted that an analysis with fixed frequency, but with the fundamentals growing in the streamwise direction, could be done in a similar way to the analysis of this paper, following Watson[6]; in that case the two streamwise wave numbers would be different.

For another treatment of interacting three-dimensional disturbances, with special reference to a resonance phenomenon, the reader is referred to a paper by Raetz[16].

As a final point, it may be worth mentioning that the "critical-layer" analysis of Lin's Freiburg paper[15] has no application within the framework of the perturbation theories of the various kinds described in refs. 1–4, and in this paper, since it[15] envisages much larger disturbances. For further discussion of this point, see ref. 3.

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